Product of two negative numbers:An example of how rote learning a strategy is synonymous with learning the concept.

Lupe M.T. Gates

University of Tasmania at Launceston

Abstract

The learning and understanding of negative numbers was part of a broader study carried out to explore the quality of mathematical thought processes (MTP) of 110 first-year mathematics university students. A mathematical item on finding the 'product of two negative numbers' was designed specifically for this purpose. The quality of the responses was assessed using the SOLO taxonomy evaluation technique (Biggs & Collis, 1982). Quality is here defined as logic, depth and clarity of MTP. An alarming 94% of the responses equated understanding to learning the 'rule', two negatives make a plus. A 71% of the responses were categorised as uni-structural SOLO level in the conrete symbolic mode. The findings strongly suggest that such learning and knowledge is a function of teaching strategy and that rote learning rules inhibits quality learning.

Introduction

The purpose of this exploration of first year university students' MTP was to gain insights into how these students, successful achievers of mathematics during pre-tertiary schooling, formulate their understanding of concepts such as negative numbers. According to Marton (1988), 'an alternative way of thinking about learning is to realise that what is learned (outcome or the result) and how it is learned (the act of the responses) are two inseparable aspects of learning'(p.53). Using this notion as a framework, one of the aims of this exploration is to identify strategies of learning and understanding which promote quality learning. Quality is here defined as logic, depth and clarity of the response.

Steefland's (1993) literature review of negative numbers showed that negative numbers have been viewed as both concrete numbers (eg. debt, loss) and as formal (algebraic) mathematical constructs and that such views have added to the confusion and difficulty of understanding the concept of negative numbers. An important issue to capitalise on from this literature review is not how educators perceive the concept but rather their strategies of delivering these perceptions of the concept that is of fundamental importance to this study. It was hypothesized that teaching strategies of finding the *product of two negative numbers* would be strongly reflected by the university students responses.

Studies on teaching strategies for integers have suggested the use of concrete materials such as bingo chips (Lytle,1994) and card/counter (Hayes,1994). The most commonly used teaching strategy for integers is the *number line* but according to Kuchemann (1981) this should be abandoned.

Context

To see which strategies are and have been promoted as teaching strategies for negative numbers one needs to look at teaching resources (eg. textbooks & videos). For example, an earlier publication of *Algebra Texts* by the Education Department of Western Australia (1977) viewed negative numbers as opposites of positive numbers. The number line approach was used as a visual representation for the set of all real numbers and the method of teaching arithmetic operations (+,-,÷,x) using negative numbers. The number line has also been widely used as a visual representation of 'directed numbers' (eg. Lynch & Parr, 1982; McSeveny et al. 1990). Hodgson and Leigh-Lancaster (1990) presented negative numbers as the inverse of natural numbers such that -1 + 1 = 0.

The teaching resources mentioned above were for high schools or secondary levels and hence, tend towards formal strategies. Some textbook writers have tried to mix both the concrete and formal representation. For example, in *Signpost Mathematics 8* by McSeveny et al. (1990), directed numbers, defined as the set of positive & negative numbers, are first introduced as loss and gain where loss is the negative number and gain is the positive number. This presentation was quickly replaced, in the very next page of the chapter, by the number line to suggest that the number line was held to be the higher order (formal) method of the two. Such an approach tends to imply that when students learn through the number line method, they have somewhat made the transition from learning negative numbers as concrete numbers to formal or abstract forms of learning.

This approach appears to have its foundations on the Piagetian cognitive developmental theory. According to Schminke, et al. (1978) the number line can be used as an excellent model for testing children on whether they have made the transition from concrete operational stage to the formal Piagetian stage with regards to our number system. This is because the number line is the symbolic abstract form of counting and ordering of numbers. It also represents numbers as measurements of distance, direction and movement (forward & reverse). Unfortunately, many mathematics textbook writers and teachers do not appreciate that children do not develop understanding overnight (or over the page!) but need time and appropriate help in developing acquired thought processes.

Rathmell's (1980) investigation of how children between the ages of 9 and 13 years relate the models of arithmetic operations to their understanding of the operations, showed that the *number line* model provided little assistance for students in understanding the operation involved, even for addition. The conclusion was that 'students need to develop a better understanding of these models, including the number line, in order to become better problem solvers' (p37). It seems that care should be exercised in adopting any particular teaching strategy, for what might be thought of as teaching the concept of negative number may in fact have been the teaching and learning of methods such as the number line rather than the concept itself.

From Rathmell (1980) and other studies (eg. Carpenter et al., 1981), it appears that the wealth of teaching ideas or models could produce and generate their own sets of concepts that students must come to grips with first, before understanding the concept(s) being taught. In effect, maths teachers, unknowingly, when using these models or strategies are providing students with problems to be solved, learning the strategy, rather than using the models as representations of the concept. Furthermore, the various teaching approaches or models used could well be considered as 'context cues'.

Bastick (1993) claims that for mathematics education, context cues are implicit and not directly transferable out of that teacher's classroom into a new context where another teacher might be advocating different strategies. He added that context cues can 'severely restrict our children's mathematical thinking because they become dependent on these 'context cues' in the maths classroom and the children cannot independently initiate them for themselves or use them out of classroom' (p87).

If textbook methods and other teaching resource methods are considered to be context cues, then it is even more important to find a teaching strategy for negative numbers which promotes learning for quality understanding as well as allowing the learning to be transferable to other contexts.

Identifying strategies of learning and understanding which promote quality learning of mathematics is the purpose of this study. This part of the study explores how University students, successful achievers of mathematics during pre-tertiary schooling, formulate their understanding of negative numbers, in particular the multiplication of negative numbers. From these University students' responses, teaching strategies used for multiplication of negative numbers could be isolated.

Methodology

To address the research issues mentioned above, the following mathematical item was used for this purpose.

A High school student was asked the following question:

What is the product of -4 and -30?

Student's response: "120 equals 4 times 30
Two negatives
make a plus"

Q1: Would you say this student understands how to find the products of negative numbers?

Q2: How would you explain the result of 120?

The above item was the *result* from responses to a mathematical item given to 23 pre-service secondary mathematics teachers who were in their second, third, and fourth year of University studies. The initial item: *How would you explain -4 X -30 to a High school student?*, was given to these students to determine their mathematical understanding of negative numbers in relation to the multiplication operation. A significant proportion of the 23 pre-service teachers responded with explanations similar to that used in the mathematical item for this study. These prospective secondary mathematics teachers were certified to have been successful in mathematics both in their pre-tertiary and tertiary studies. Rote learning the rules was the teaching strategy for multiplication of negative numbers that emerged from their responses. *Is this how other high achievers of mathematics from pre-tertiary also understand and have learnt multiplication of negative numbers?* This was the question that was explored in this part of the study.

The qualitative nature of the 'mathematical item' as presented above requires a categorisation system that takes account of the quality of the response. The SOLO taxonomy was used for this purpose and according to Marton (1988) SOLO technique is appropriate 'for describing the relationship between approach [to learning] and outcome' (p.70). For more detail on the SOLO technique see Biggs and Collis (1982).

Sample

The mathematical item was given to 110 first-year university students in their third week at University. These students were enrolled in the first compulsory unit of mathematics for study programs with high mathematical requirements such as Engineering, Applied Science & Technology and Maths & Science Education. The sample included mature age students who have completed mathematical entrance requirements or have completed a degree/diploma with a strong mathematical background such as Diploma of Engineering. The sample also included overseas students (10%) as well as a representative number from all the Australian states. The sample distribution was not pre-planned. However, the key criterion with the sample is that these students have been classified and certified as capable students of mathematics. Most of the sample are enrolled in the Bachelor of Engineering program.

Results & Discussions

Ninety-four percent (94%) of the responses said 'yes' to question one indicating that knowing "two negatives make a plus" is synonmous with understanding of how to find, compute, the products of negative numbers. Seventy-one percent (71%) of responses were classified into uni-structural SOLO level and nineteen percent (19%) in the multi-structural level of the concrete symbolic mode. The following are representative of the common responses:

Q1: Yes
Q2: -4 x -30=120; Rules when product or division: -ve x +ve = -ve, -ve x -ve = +ve, +ve x +ve = +ve.
Q1: Not necessarily Q2: -4 x -30 = 120
Q1: Yes
Q2: Same as student
Q1: Yes
Q2: No different to the student

These responses were classified as uni-structural in the concrete mode in that knowledge elements are routine learnings of a rule. The following response is representative of a multi-structural response in the concrete mode where in additional to routine learnings of a rule it draws attention to other associated knowledge elements of the concept.

Q1: Yes
Q2: -4 x -30 = 120. Two negatives combine to give a positive.Similar to subtracting a negative amount.
Eg. 4-(-3)=7 if you are taking away a negative it is the same as adding a positive.

The high frequency of responses in these low SOLO levels was a concern because this 'mathematical item' was classified as the 'easy item' in comparison to the other three items in the main study and in relation to the conceptual understanding involved. Accordingly responses to this item should be at least a multi-structural SOLO level. It was then thought that the 'item' was not allowing the respondents to demonstrate accurately their understanding. In other words, the respondents know more but the *format* and *wording* of the item was inhibiting true levels of understanding. To explore this further a small sample of 15 was selected randomly from the 71% pool of responses and these were interviewed to elaborate on their initial responses. The following are extracts from some of these interviews:

Interview question: Could you elaborate on your response?

"Just rules I have learnt, I don't know why though. There's a lot of maths that doesn't make a lot of sense to me & this is one of them ..."

"I know I haven't answered the question but I don't know why. I can't recall,that's about 30 yrs ago, ever been explained why the product of two negatives is a plus. I just learnt it that way and I never questioned why, I just accept it as is..."

"I just agreed with the student's response as I can't explain it any differently..but can you prove such a thing mathematically? Perhaps you could show using blocks. Still I don't know how you could translate it to mathematics. I guess that's why teachers don't give a proof of the rules for negatives because there is none..."

"Yeh, well, sorry I don't know really. Sorry, I just don't know how. If I was shown how I just can't remember it now..."

"I don't know why when you multiply two negatives give a positive, I just know it that

> way. There could be a complicated proof but why would one want to waste time proving something like this. Probably maths teachers must know how to prove it."
>
> "No. I can't see the problem so there's little to explain."

The interviews have shown that the 'item' was not inhibiting the respondents to demonstrate their knowledge and understanding of the subject matter in question. Rather, there appears to be a clear lack of depth of knowledge. Also the 'lack of memory' to recall lengthy explanations given years ago appears to be a contributing factor for knowing just the rules. The disturbing factor from these results and interviews is the strong indication that the teaching strategy used was strictly rote learning the 'rules'. Another disturbing factor is that 'place of schooling', where the respondents received pre-tertiary schooling, has no bearing. That is, similar strategies, rote learning 'rules', appear to be highly favoured as methods of teaching, in particular how to find products of negative numbers, throughout the areas represented in the sample. Rote learned rules and procedures were also reported by Eisenhart et al. (1993) as favoured methods of teaching mathematics in secondary schools covered by their study.

To explore further the 'emerged notion' that the favoured teaching strategy for learning multiplication of negative numbers is just 'rote learning rules', a sample of 20 college (years 11 & 12) mathematics teachers was presented with the same mathematical item as that given to the university students. The teachers were from some of the main feeder colleges to the University and some of the university students involved in the study have been taught by several of these teachers. These teachers were asked to respond to the item as well as commenting on how they have (would have) taught negative numbers and in particular the product of two negative numbers.

The responses from the teachers showed a strong consensus that *knowing the rules is satisfactory* particularly when the students are bright (more able) since these students need more time to spend on the learning of other more important areas of the mathematics curriculum. Responses for teaching strategies were strongly in favour of the 'number line' as a method of *introducing the concept of negative numbers* particularly at the lower high school levels. These responses appear to reflect the methods of introductions for negative numbers used in Mathematics textbooks (eg. Lynch & Parr,1982; Hodgson & Leigh-Lancaster,1990; McSeveny et.al, 1990) that are commonly used in secondary levels.

However, when teachers were asked how 'multiplication' is shown on the 'number line' the common response was *repeated additions, but it has been a long time since I've taught it, but that's how I'll explain it.* Teachers who have taught in high schools before college level were more willing to provide alternative methods, for example, *negative numbers can be represented as amounts or quantities 'borrowed & owed.* However, there were no clear and simple algorithms shown from these teachers' responses apart from an algebraic approach as follows:

-4 x -30 = (-1)4 x (-1)30 = (-1)(-1) x 4(30) = 120. Again this algorithm still relies on the learner knowing the rule for (-1)(-1)=1.

From these teachers' responses it seems that the university students' knowledge of negative numbers, particularly finding products of negative numbers, is a function of the way they have been taught, rote learning rules. There is also the suggestion that in trying to provide a less confusing approach to learning negative numbers to give the learners some consistency, the learning of rules appeared to be the favoured, considered appropriate, method adopted by teachers for teaching able students of mathematics at secondary level.

Similar results were shown by Eisenhart et al. (1993) from their study of educators, classroom teachers and pre-service teachers of mathematics. These authors have shown that although mathematics teachers and pre-service teachers were shown to be well aware of the value and the importance to teach for conceptual understanding, in practice these teachers taught using procedural or rote learning methods. As for the use of the number line as a teaching strategy and a model for negative numbers, it appears that it is still the favoured method for introduction.

Implications & Conclusion

These qualitative results provide an insight as to how students, successful achievers of mathematics during pre-tertiary schooling, view and learn negative numbers, particularly multiplication of negative numbers. The results showed that teaching strategies, rote learning rules, employed by teachers reflected strongly in students' responses. This finding suggests that teaching strategies used for learning have a strong influence on students' learning and may also provide determinants for the quality of mathematical thought processes (MTP) inherent by students of mathematics. The results also seemed to indicate that the adoption of rote learning rules by students are directly related to the absence or lack thereof of appropriate models for learning the construct, multiplication of negative numbers. However, according to Shulman (1987, p.16-17), the strength of models, representations of mathematical concepts, still lies with the teachers ability to "transform" the content and make it comprehensible and appropriate to the ability levels and interests of learners.

The high frequency of 'yes' responses and uni-structural SOLO level responses suggested that rote learning of the rules is synonymous to understanding. This is a concern because it reflects a 'lazy attitude' to learning which does not promote problem solving and independent thinking that are essential thought processes for higher learning which typifies University learning.

The findings of this study tentatively suggest that teachers remove from their students the 'power' or initiative to learn for quality when employing rote learning teaching strategies. This view is also shared by Greenwood (1993) who claims that 'mathematical power is gained by minimising the student's dependence on the teacher or answer key' (p.144). Thus, being dependent on a rule ,*answer key,* could reduce the inner power of a student to learn.

For able students of mathematics it appears that teaching strategies that save time and easy to apply, presumably easy to learn, for a concept that appears trivial, such as negative numbers, do not develop depth and clarity of understanding of the concept beyond what and how it was taught. In other words, rote learning rules type teaching strategies inhibit learning for quality.

Another notion emerging from the findings is that able students of mathematics are much better at memorising and recalling methods that provide quick answers than methods which require in depth explanations. For example, methods such as the use of concrete models and manipulation of materials as suggested by Reys et al. (1992) may require in depth explanations by the learner later on. If these university students did receive learning of the concept, multiplying two negative numbers, by means of activities and manipulation of concrete materials earlier in High school and later learned the 'rule', then from the results of this study the former strategy had very little affect on students understanding of the concept. Consequently it appears that rote learning of rules override other learning methods.

It is reasonable to suggest that able students of mathematics are good 'organisers' of their learnings such that they can successfully apply these learnings even if the meanings are not explicitly understood. Nevertheless, if students, whether they have high or low abilities, are to be taught mathematics it is important to employ teaching strategies that promote quality learnings which are not only transferable to other contexts but can be integrated with others.

A couple of questions that have emerged from these students' responses, although not the focus of this study but worthy of investigation are: how crucial are the understandings of 'trivials' (eg. product of negative numbers) in the development of quality mathematical thought processes and how can earlier conceptual learnings through concrete models be transferable and integrated to abstract higher order learnings. Clearly the answers relating to these questions would greatly assist our knowledge of better teaching strategies that promote quality learning of mathematics.

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